# Ansatzfunktionen und deren Ableitungen

Approximation der Knotenwerte im Element durch Ansatzfunktion:
($\xi, \eta$ : isoparam. Koordinaten)
mit $N_a = N_a(\xi, \eta)$

$$\boldsymbol{x} = \sum_a^{\text{nel}} N_a \cdot \boldsymbol{x}_a^e \tag{1}$$

$$\boldsymbol{u} = \sum_a^{\text{nel}} N_a \cdot \boldsymbol{u}_a^e \tag{2}$$

$$\boldsymbol{u}_{,x} = \sum_a^{\text{nel}} N_{a,x} \cdot \boldsymbol{u}_a^e \tag{3}$$

$$\boldsymbol{u}_{,y} = \sum_a^{\text{nel}} N_{a,y} \cdot \boldsymbol{u}_a^e \tag{4}$$

$\Longrightarrow$

$$N_{a,x} = N_{a,\xi} \cdot \frac{\partial \xi}{\partial x} + N_{a,\eta} \cdot \frac{\partial \eta}{\partial x} \tag{5}$$

$$N_{a,y} = N_{a,\xi} \cdot \frac{\partial \xi}{\partial y} + N_{a,\eta} \cdot \frac{\partial \eta}{\partial y} \tag{6}$$

$\Longleftrightarrow$

$$\begin{bmatrix} N_{a,x} \\ N_{a,y} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix}}_{=\left(\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{x}}\right)^T} \begin{bmatrix} N_{a,\xi} \\ N_{a,\eta} \end{bmatrix} \tag{7}$$

mit

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad \boldsymbol{\xi} = \begin{bmatrix} \xi \\ \eta \end{bmatrix} \tag{8}$$

$$\Rightarrow \quad \left(\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{x}}\right)^T = \frac{\partial \xi_j}{\partial x_i} \vec{e}_i \otimes \vec{e}_j \tag{9}$$

$\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{x}}$ ist nicht bekannt, es kann aber $\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}}$ berechnet werden:

$$\frac{\partial x}{\partial \xi} = \sum_a^{\text{nel}} N_{a,\xi} x_a^e \tag{10}$$

$$\frac{\partial x}{\partial \eta} = \sum_a^{\text{nel}} N_{a,\eta} x_a^e \tag{11}$$

$$\frac{\partial y}{\partial \xi} = \sum_a^{\text{nel}} N_{a,\xi} y_a^e \tag{12}$$

$$\frac{\partial y}{\partial \eta} = \sum_a^{\text{nel}} N_{a,\eta} y_a^e \tag{13}$$

$$\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} \tag{14}$$

$$\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}}\right)^T = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \tag{15}$$

$$\Rightarrow \left(\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{x}}\right)^T = \left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}}\right)^{-T} = \frac{1}{\det\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}}\right)} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{bmatrix} \tag{16}$$

$$\Rightarrow \begin{bmatrix} N_{a,x} \\ N_{a,y} \end{bmatrix} = \frac{1}{\det\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}}\right)} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{bmatrix} \begin{bmatrix} N_{a,\xi} \\ N_{a,\eta} \end{bmatrix} \tag{17}$$

FEM-Notation: (Bsp. lineares Vierknotenelement)

$$\Rightarrow \text{shape} = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix}_{\text{akt.GP}} \tag{18}$$

Ableitungen nach $\boldsymbol{\xi}$:

$$\text{nshape} = \begin{bmatrix} N_{1,\xi} & N_{1,\eta} \\ \vdots & \vdots \\ N_{n,\xi} & N_{n,\eta} \end{bmatrix}_{\text{akt.GP}} \tag{19}$$

$$\boldsymbol{x}^e = \begin{bmatrix} x_1 & y_1 \\ \vdots & \vdots \\ x_4 & y_4 \end{bmatrix} \tag{20}$$

$$\boldsymbol{u}^e = \begin{bmatrix} u_{x1} & u_{y1} \\ \vdots & \vdots \\ u_{x4} & u_{y4} \end{bmatrix} \tag{21}$$

$$\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}}\right) = \boldsymbol{x}^{e^T} \cdot \text{nshape} = \begin{bmatrix} \cdots & x & \cdots \\ \cdots & y & \cdots \end{bmatrix} \begin{bmatrix} \vdots & \vdots \\ N_{,\xi} & N_{,\eta} \\ \vdots & \vdots \end{bmatrix} \tag{22}$$

$$\left(\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{x}}\right) = \text{inv}\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}}\right) \tag{23}$$

$$j = \det\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}}\right) \tag{24}$$

$$\text{dshape} = \text{nshape} \cdot \left(\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{x}}\right) = \begin{bmatrix} \vdots & \vdots \\ N_{,\xi} & N_{,\eta} \\ \vdots & \vdots \end{bmatrix} \cdot \begin{bmatrix} \xi_{,x} & \xi_{,y} \\ \eta_{,x} & \eta_{,y} \end{bmatrix} \tag{25}$$